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ON THE PROBLEM OF IMPRESSING A THIN RIGID BODY INTO A MEDIUM WITH HARDENING

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The boundary separating the plastic and the rigid domains is determined on the basis of the linearized solution [1] of the problem of impressing a thin rigid body into a plastic medium possessing translational hardening. The case of a solid in the shape of a wedge is considered. In particular, the solution of the problem of impressing a thin wedge into an ideally plastic half-space is obtained when hardening is neglected; a comparison is made with the known solution of Hill, Lee and Tupper [2].

1. Considering a plastic material to be under plane strain conditions, and directing the coordinate axes as shown in Fig. 1a, let us write the equation of the solid surface as

$$y = \delta f(x), \quad f(0) = 0, \qquad F_2(0) = 0 \ (F_i \equiv d^i f \mid dx^i) \tag{1.1}$$

where δ is a small dimensionless parameter, and f is a sufficiently smooth function. The material occupies the half-space $x \leqslant 0$ at the initial instant. Reversing the motion, let us consider the solid fixed and the medium to move translationally upward along the x-axis at some constant velocity.

Let us henceforth use the variables

$$\eta = x - y, \quad \xi = x + y \tag{1.2}$$

in addition to the variables x, y.

The linearized solution of the problems has been found in [1]. It follows therefrom that the plastic domain AOB (Fig. 2) consists of two zones: OBC $(0 \leqslant \xi \leqslant h)$ and ABC $(h \leqslant \xi \leqslant 2h)$. The stresses on the line BC $(\xi = h)$ are continuous. The equation of the buckling surface of the plastic material is [1]

$$x - h = \delta f(h - y) \tag{1.3}$$

In a zero approximation the boundary separating the plastic and rigid domains is defined by the equation $x - y = 0$ (Fig. 1b), which in the η, ξ variables has the following form:

$$\eta(\xi) = 0 \tag{1.4}$$

Varying (1.4), we obtain the equation of the rigid-plastic boundary in the first approximation $\eta + \delta\gamma'(\xi) = 0$ or in the x, y variables

$$x - y + \delta\gamma'(x + y) = 0 \tag{1.5}$$

Here the function $\gamma'(x + y) = \gamma'(\xi)$ is to be determined.

The rigid-plastic boundary in the problem under consideration is a slip line of the second family (β-line). Hence its differential equation is

$$dy / dx = -\operatorname{ctg}\theta \tag{1.6}$$

where the meaning of the angle θ is seen from Fig. 3. Let us evaluate the derivative dy/dx by means of (1.5). Then (1.6) becomes

$$\operatorname{ctg}\theta = -[1 + 2\delta\Gamma'(\xi)], \qquad \Gamma'(\xi) = d\gamma'(\xi)/d\xi \tag{1.7}$$

The first principal direction encloses the angle

$$\operatorname{tg}2\,(1,\ x) = 2\tau/\sigma_x - \sigma_y \tag{1.8}$$

with the x-axis (Fig. 3). It is seen that $(1, x) = \theta + \pi/4$, hence $\operatorname{tg}2\,(1,x) = -\operatorname{ctg}2\theta$. Comparing with (1.8), we find

$$2\tau/\sigma_x - \sigma_y = -\operatorname{ctg}2\theta \tag{1.9}$$

We evaluate $\operatorname{ctg}2\theta$ by means of (1.7) and insert the result into (1.9). We then have the relationship

$$\frac{\delta\tau'}{2k + \delta(\sigma_x' - \sigma_y')} = \delta\Gamma'(\xi)$$

upon linearizing it we obtain

$$\Gamma'(\xi) = \tau'/2k \quad \text{for} \quad \eta = 0 \tag{1.10}$$

Integrating (1.10) in the zone OBC according to [1] yields

$$\gamma'(\xi) = \frac{1}{2k}\left\{\left(k - \frac{c}{2}\right)\xi F_1(0) + \left(k + \frac{c}{2}\right)f(\xi) - \operatorname{ch} F_1(\xi) + c\,[\xi F_1(\xi) - f(\xi)]\right\} + a \tag{1.11}$$

$$a = c\,\frac{h}{2k}\,F_1(0)$$

The constant of integration a is found from the condition that the rigid-plastic boundary passes through the origin. Inserting (1.11) into (1.5), we finally obtain for the zone OBC

$$x - y + \delta/2k\,\{(k - \tfrac{c}{2})\,[f(x + y) + (x + y)\,F_1(0)] + c\,[hF_1(0) + (x + y - h)\,F_1(x + y)] = 0 \tag{1.12}$$

Integrating (1.10) in the zone ABC, we obtain

$$\gamma'(\xi) = \frac{1}{2k}\left[\left(k - \frac{c}{2}\right)\xi F_1(0) - \left(k + \frac{c}{2}\right)f(2h - \xi)\right] + b \tag{1.13}$$

Here the constant of integration b cannot be determined from the condition that the rigid-plastic boundary passes through the point P at which the material of the medium starts to buckle (Fig. 1a) since only the abscissa of this point $x_p = h$, is known, but not its ordinate. In fact, every point with coordinates $(h,\ h + \delta\lambda)$, where λ is an arbitrary real number, satisfies (1.3) of the buckling surface of the medium. Hence, we shall seek the constant of integration b from the condition that the rigid-plastic boundary passes through the point of intersection N of this boundary with the line QN separating the plastic zones in the physical plane (Fig. 1a). We find the equation of the line QN as follows. In a zero-th approximation QN coincides with the line BC (Fig. 2) and is

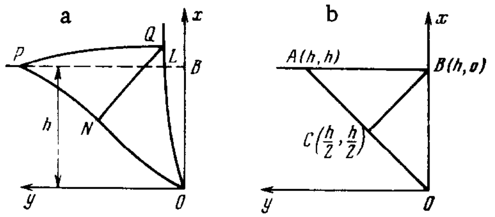

Fig. 1

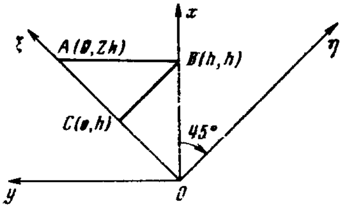

Fig. 2

described by the equation $\xi - h = 0$. Varying and going over to the x, y coordinates, we obtain the equation of the line QN

$$x + y - h + \delta\omega\,(x - y) = 0 \tag{1.14}$$

The line QN is a first family slip line (α-line), hence its differential equation is

$$dy \,/\, dx = \operatorname{tg}\theta \tag{1.15}$$

Evaluating the derivative $dy\,/\,dx$ by means of (1.14) and substituting the result into (1.15), we rewrite it as

$$\operatorname{tg}\theta = -\,(1 + 2\delta d\omega \,/\, d\eta) \tag{1.16}$$

The relationship (1.9), taking account of (1.16), becomes

$$\frac{\tau}{\sigma_x - \sigma_y} = -\delta\frac{d\omega}{d\eta} \tag{1.17}$$

Inserting the stress into (1.17) and linearizing, we obtain

$$\frac{d\omega}{d\eta} = -\frac{\tau'}{2k} \quad \text{for} \quad \xi = h \tag{1.18}$$

Integrating this relationship yields (A is the constant of integration)

$$\omega(\eta) = -\frac{1}{2k}\left[kf(\eta)\left(k + \frac{c}{2}\right)\eta F_1(h) + \frac{c}{2}(h - \eta)\,F_1(\eta)\right] + A \tag{1.19}$$

Substituting (1.19) into (1.14) for the line QN we have

$$x + y - h - \frac{\delta}{2k}\left\{-2kA + \left[kf(x - y) + \left(k + \frac{c}{2}\right)(x - y)\,F_1(h) - \frac{c}{2}(x - y - h)\,F_1(x - y)\right]\right\} = 0 \tag{1.20}$$

Solving (1.1) and (1.3) jointly, we find the coordinates of the point Q (Fig. 1a)

$$x_Q = h + \delta f(h), \quad y_Q = y_L = \delta f(h) \tag{1.21}$$

Substituting the coordinates of the point Q from (1.21) into (1.20) for the line QN, we find the value of the constant A. Then (1.20) for the line QN becomes

$$x + y - h - \frac{\delta}{2k}\left[kf(x - y) + 3kf(h) + \left(k + \frac{c}{2}\right)(x - y - h)\,F_1(h) - \frac{c}{2}(x - y - h)\,F_1(x - y)\right] = 0 \tag{1.22}$$

We find the coordinates of the point N by solving (1.22) for the line QN jointly with (1.12) for the line ON (Fig. 1a)

$$x_N = \frac{h}{2} + \frac{\delta}{4k}\left\{2kf(h) - kh\,[F_1(0) + F_1(h)] + \frac{c}{2}\,[f(h) - hF_1(h)]\right\}$$
$$y_N = \frac{h}{2} + \frac{\delta}{4k}\left\{4kf(h) + (k + c)\,h\,[F_1(0) - F_1(h)] - \frac{c}{2}\,[f(h) - hF_1(h)]\right\} \qquad (1.23)$$

Now, the remaining section NP of the rigid-plastic boundary can be found. According to (1.5) and (1.13) its equation is

$$x - y + \frac{\delta}{2k}\left[\left(k - \frac{c}{2}\right)(x + y)\,F_1(0) - \left(k + \frac{c}{2}\right) f\,(2h - x - y) + 2kb\right] = 0 \qquad (1.24)$$

Substituting the coordinates of the point N from (1.23) into (1.24), we find the value of the constant b. Then (1.24) for the line NP becomes

$$x - y + \frac{\delta}{2k}\left[2kf(h) + \left(k - \frac{c}{2}\right)(x + y)\,F_1(0) - \right. \qquad (1.25)$$
$$\left.\left(k + \frac{c}{2}\right) f\,(2h - x - y) + \mathrm{ch}\ F_1(0)\right] = 0$$

Therefore, the rigid-plastic boundary is defined completely by the relationships (1.12) and (1.25), respectively, for the zones OBC and ABC. Setting $x = h$ in (1.25), we find the ordinate of the point P

$$y_P = h + \delta\,[f(h) + hF_1(0)] \qquad (1.26)$$

Let us note that by virtue of the incompressibility of the material of the medium, the areas of the curved figures OLB and PQL (Fig. 1a) should be equal. It is seen that this integral relationship is satisfied identically; the total value of the mentioned areas turns out to equal

$$\delta\left[\frac{F_1(0)}{2!}h^2 + \frac{F_3(0)}{4!}h^4 + \frac{F_4(0)}{5!}h^5 + \ldots\right]$$

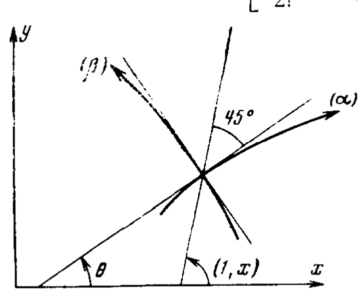

Fig. 3

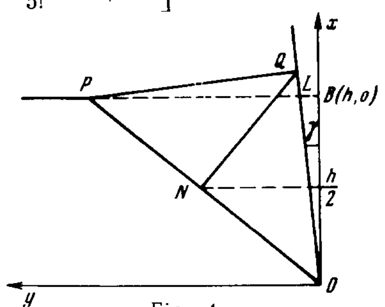

Fig. 4

2. Let us consider a solid in the shape of a wedge. In this particular case the function f in (1.1) is linear, and it becomes

$$y = (\mathrm{tg}\alpha)x \quad (\mathrm{tg}\alpha = \delta F_1(0)) \qquad (2.1)$$

Here α is the half-angle of the wedge vertex (Fig. 4). According to [1], the complete linearized solution is

$$u = u^\circ + \delta u^\circ F_1(0), \quad v = \delta u^\circ F_1(0)$$
$$s_x = h + \delta F_1(0)\,(x - y), \quad s_y = \delta F_1(0)\,(x - y) \qquad (2.2)$$
$$e_x = -e_y = \delta F_1(0), \quad e_{xy} = 0$$
$$\sigma_x = 0, \quad \tau = 2\delta k F_1(0), \quad \sigma_y = -2k - 2\delta c F_1(0)$$

Let us note that each of the stresses is given by one formula in the whole plastic domain. Moreover, the stresses are constant, therefore, the state of stress is homogeneous everywhere in the plastic domain.

The equation of the buckling surface of the medium is

$$x - h = \delta F_1(0)(h - y) \tag{2.3}$$

by virtue of (1.3) and (2.1). According to (1.12), (1.25) and (2.1), the rigid-plastic boundary is given by the equation

$$x - y + \delta F_1(0)(x + y) = 0 \tag{2.4}$$

The relationships (2.3) and (2.4) show that the boundary of the buckling material and the rigid-plastic boundary are rectilinear (Fig. 4). The coordinates of the point N are P are found from (1.23), (1.26) and (2.1). According to (1.21) and (2.1) the point Q has the coordinates

$$x_Q = h + \delta h F_1(0), \quad y_Q = y_L = \delta h F_1(0)$$

According to (1.22) and (2.1) the line QN is

$$x + y - h - \delta F_1(0)(x - y + h) = 0$$

In the case of an ideally plastic medium, $c = 0$ in all the formulas in Sect. 1. Moreover, if the boundary of the solid is rectilinear, then all the formulas in Sect. 2, with the exception of the last formula for the stress σ_y in (2.2), are valid for an ideally plastic medium since they do not contain the parameter c. The stress σ_y becomes

$$\sigma_y = -2k \tag{2.5}$$

It is interesting to compare the solution obtained with the known solution of Hill, Lee and Tupper for the problem of impressing a wedge in an ideally plastic half-space [2]. In particular, it is seen that a centered fan with its vertex angle β defined by the formula

$$2\alpha = \beta + \operatorname{arccostg}(\pi/4 - \beta/2) \tag{2.6}$$

is still located between the triangles PQN and OQN (Fig. 4). If the vertex angle 2α of the wedge is small, as in the case considered here, then the vertex angle β of the fan is also small. It then follows from (2.6) that to higher order accuracy, the equality $\beta = 2\alpha^2$ is satisfied, which we rewrite as

$$\beta = 2\delta^2 F^2_1(0) \tag{2.7}$$

As is seen from (2.7), the fan appears only in the second approximation; in the first approximation it degenerates into the line QN (Fig. 4).

Therefore, the geometric picture of impression of a wedge according to Hill agrees with that represented in Fig. 4. The stress fields also agree in both cases. Regarding the velocity fields, they are identical to the accuracy of the reversal of the motion.

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